Advanced Quantum Mechanics UE
WS 2014/15 1.st test, Jan 26, 2015

- **Please start each exercise on a new sheet of paper. Don't forget to write your name and Matrikelnnumer on each sheet**
- **You are allowed to use just the two Quantum Mechanics lecture notes, two A4 pages with your own notes, paper, pen, a pocket calculator. Please put cellphones, books, other written material in your bag**
- $\hbar = 1$ **everywhere**

1. Time-dependent perturbation theory (14P=4+2+3+2+3)

The bound states $|\varphi_{\ell,m}\rangle$ of a quantum mechanical particle in a rotation invariant potential have energies $E_{\ell,m} = q\ \ell$. Their unnormalized wave functions are given by

$$\langle \boldsymbol{r}|\varphi_{\ell,m}\rangle = Y_{\ell,m}(\theta,\varphi)\ \frac{e^{-\alpha\ r/2}}{r} \ .$$

Here, $Y_{\ell,m}$ are the spherical harmonics, and α, q are positive constants. The particle is initially in the ground state $\ell = m = 0$.

(A) Consider a time-depend perturbation ($\gamma > 0$)

$$\hat{W}(t) = e^{\gamma t}\ \hat{V}_0 \ ,$$

which acts from $t = -\infty$ to $t = 0$. Write down the probability of transition at $t = 0$ to a generic excited state $|\varphi_{\ell,m}\rangle$ in first order perturbation theory (the operator $\hat{V}_0$ is still unknown: just express the result in terms of matrix elements but carry out the time integrals)

(B) Consider the case

$$\hat{V}_0 = B\ \hat{z} \ .$$

For which final state(s) $|\varphi_{\ell,m}\rangle$ is this probability nonzero?

(C) Determine this probability.

(D,E) Same questions as in (2) for the case

$$\hat{V}_0 = B\ \hat{z}^2 \ .$$

Hint: $\int_0^\infty r^n e^{-Ar} dr = n!\ A^{-(n+1)}$ $\qquad \int_{-\infty}^0 e^{Ax}\ dx = 1/A$

You don't need to evaluate the angular integrals: multiply z, resp. z^2 in polar coordinates by $Y_{0,0}$ and expand the result in terms of the $Y_{\ell,m}$ (look at the tables in the lecture notes).

2. Two spin-1 particles (13P=2+3+2+2+2+2)

Two spin-1 particles $i = 1, 2$ interact with the hamiltonian

$$\hat{H} = \lambda\ \hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_2 \qquad \text{(scalar product)} \ ,$$

Please turn over

where $\hat{\boldsymbol{S}}_i$ is the spin operator of particle i, and λ a positive constant.
(A) Let $\hat{\boldsymbol{J}} = \hat{\boldsymbol{S}}_1 + \hat{\boldsymbol{S}}_2$ be the total spin. What are the possible values of $\hat{\boldsymbol{J}}^2$?
(B) Determine the eigenvalues of $\hat{H}$ as well as their degeneracies.
(C) The basis states of the two-spins system can be expressed, on the one hand, as tensor products

$$|m_1\rangle\,|m_2\rangle\ , \tag{1}$$

where $m_i = -1, 0, 1$ are the values of $\hat{\boldsymbol{S}}_{iz}$, the z component of the spin of particle i. One the other hand, we denote as $|j, m\rangle$ states with a well defined value of $\hat{\boldsymbol{J}}^2$ and $\hat{\boldsymbol{J}}_z$.
Express the state $|j = 2, m = -2\rangle$ in terms of the product states (1).
(D) Express the state $|j = 1, m = 1\rangle$ in terms of the product states (1).
(E) Evaluate the expectation value of $\hat{\boldsymbol{S}}_{1z}$ in $|j = 1, m = 1\rangle$. What is the probability that a measure of $\hat{\boldsymbol{S}}_{1z}$ in this state gives 0 ?
(**backup:** If you haven't solved (D) use $|j = 1, m = 1\rangle \to |-1\rangle\,|0\rangle + |0\rangle\,|-1\rangle$)
(F) Express the state $|j = 1, m = 0\rangle$ in terms of the product states (1).

3. Two bosons in an harmonic oscillator (13P=2+2+2+2+3+2)

Consider two bosons in an harmonic oscillator with $m = \omega = \hbar = 1$.
Let $b_n^\dagger$ be the operator creating a boson in the level $|n\rangle$ of the harmonic oscillator.
(A) Write down the two-boson ground state $|G\rangle$ in second quantisation, as well as its energy.
(B) Write down the first excited state(s) $|E\rangle$, as well as their energy and degeneracy.
(C) Write down the second excited state(s), as well as their energy and degeneracy.
(D) Write down the position operator $\hat{X}$ in second quantisation.
Hint: In first quantisation

$$\langle n|\,\hat{X}\,|m\rangle = \sqrt{\frac{n}{2}}\delta_{n-1,m} + \sqrt{\frac{m}{2}}\delta_{n+1,m}$$

(E) Determine the expectation value

$$\langle E|\,\hat{X}\,|G\rangle$$

Hint.: First evaluate $\hat{X}\,|G\rangle$ and verify that it is just proportional to $|E\rangle$.
(**backup:** If you haven't solved (D) use $\hat{X} \to i\left(b_n^\dagger b_{n-1} - b_{n-1}^\dagger b_n\right)$)
(F) The system is initially in its ground state. At time $t = 0$ a perturbation

$$\hat{V} = \alpha\,\hat{X}$$

is switched on. Determine the transition probability to the excited state $|E\rangle$ as a function of time in first order perturbation theory.